WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 3rd Semester Examination, 2022-23

MTMHGEC03T/MTMGCOR03T-MATHEMATICS (GE3/DSC3)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any *five* questions: $2\times 5=10$

(a) State the Archimedean property of $\mathbb{R}$.

(b) Find the cluster points of the set

$$S=\{1,-1,1\tfrac{1}{2},-1\tfrac{1}{2},1\tfrac{1}{3},-1\tfrac{1}{3},\cdots\}.$$

(c) Find the greatest lower bound of the set $S=\left\{\frac{5}{n}:n\in \mathbb{N}\right\}$.

(d) Evaluate $\lim\limits_{n\to\infty}\left\{\frac{1^3}{n^4}+\frac{2^3}{n^4}+\frac{3^3}{n^4}+\cdots+\frac{n^3}{n^4}\right\}$.

(e) Test the convergence of the series $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$.

(f) Find the radius of convergence of

$$x+\frac{x^2}{2^2}+\frac{2!}{3^3}x^3+\frac{3!}{4^4}x^4+\cdots\cdots.$$

(g) Test the convergence of the series

$$\sum_{n=1}^{\infty}\sin\frac{1}{n}.$$

(h) Give an example of a Cauchy sequence with proper justification.

(i) Show that $\sum\limits_{n=1}^{\infty}\frac{\sin x}{n^2+n^4x^2}$ is uniformly convergent for all real x.

2. (a) If $x, y\in\mathbb{R}$ with $x>0$, $y>0$ then prove that there exists a natural number n such that $ny>x$. 4

(b) Let A be a non empty bounded above subset of $\mathbb{R}$. Let 4

$$B=\{-x:x\in A\}$$

Show that B is a non empty bounded below subset of $\mathbb{R}$ and

$$\inf B=-\sup A.$$

3. (a) Let A and B be subsets of $\mathbb{R}$ so that $A \subseteq B$. Let x be a cluster point of A. Show that x is a cluster point of B. 4

(b) Show that 1 and -1 are limit points of the set. 4

$$S=\left\{(-1)^m+\frac{1}{n} : m, n \in \mathbb{N}\right\}$$

4. (a) Justify that $\mathbb{Z}$ is a countable set. 4

(b) Show that the open interval (0, 1) is an uncountable set. 4

5. (a) Show that the sequence $\{x_n\}$ is monotone increasing, where 5

$$x_n = 1+\frac{1}{2^2}+\cdots+\frac{1}{n^2} \quad \text{for all } n \in \mathbb{N}$$

Hence show that the sequence $\{x_n\}$ is convergent.

(b) Apply Cauchy's criterion for convergence to show that the sequence $\left\{\frac{n}{n+1}\right\}$ is convergent. 3

6. (a) Show that the series 4

$$\sum_{n=1}^{\infty} \frac{3\cdot 6\cdot 9\cdots(3n)}{7\cdot 10\cdot 13\cdots(3n+4)}x^n$$

converges if $0 < x < 1$ and diverges if $x > 1$.

(b) Examine the convergence of the series 4

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{2n+1}{n(n+1)}.$$

7. (a) Show that an absolutely convergent series is convergent. 4

(b) Give an example of a convergent series which is not absolutely convergent. 1

(c) Show that the sequence $\left\{\frac{n}{n+1}\right\}$ is a Cauchy sequence. 3

8. (a) Show that the sequence of functions $\{f_n\}$, where for all $n \in \mathbb{N}$, 4

$$f_n(x) = \frac{x^n}{1+x^n}, \quad x \geq 0$$

is pointwise convergent on $[0, \infty)$ but is not uniformly convergent on $[0, \infty)$.

(b) Examine uniform convergence of the sequence of functions $\{f_n\}$ on [0, 2], where for all $n \in \mathbb{N}$, 4

$$f_n(x) = \frac{x^n}{1+x^n}, \quad x \in [0, 2].$$

9. (a) Show that the series $\sum_{n=0}^{\infty}(1-x)x^n$ is not uniformly convergent on [0, 1]. 4

(b) With proper justification, show that $\lim_{x\to 0}\sum_{k=2}^{\infty}\frac{\cos kx}{k(k+1)}=\frac{1}{2}$. 4

10. (a) Find the radius of convergence of the power series 3

$$\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 5\cdot 8\cdots(3n-1)}x^n$$

(b) Use the fact that 5

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \qquad \forall\,|x|<1$$

to obtain the power series of $\frac{1}{(1+x)^3}$.

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WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 3rd Semester Examination, 2021-22

MTMHGEC03T/MTMGCOR03T-MATHEMATICS (GE3/DSC3)

REAL ANALYSIS

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5 = 10$

(a) Find the least upper bound of the set $S=\left\{\frac{1}{p}+\frac{1}{q}\ :\ p,\, q\in\mathbb{N}\right\}$.

(b) Prove that $\mathbb{N}$ is not bounded above.

(c) Show that 0 is a cluster point of the set $S=\left\{\frac{1}{n}\ :\ n\in\mathbb{N}\right\}$.

(d) Show that $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$.

(e) Examine the convergence of the series $\sum\limits_{n=1}^{\infty}\frac{(n\,!)^2}{(2n)!}$.

(f) Examine whether the sequence of functions $\{f_n\}$ converges uniformly on $\mathbb{R}$, where for all $n\in\mathbb{N}$,

$$f_n(x)=\frac{x}{n}\ ,\ \text{for all } x\in\mathbb{R}.$$

(g) Show that the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^3+n^4x^2}$ is uniformly convergent on $\mathbb{R}$.

(h) Find the radius of convergence of the power series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n n!}{n^n}x^n$.

(i) Show that the sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence.

2. (a) If S is a non empty subset of $\mathbb{R}$ and also bounded below then prove that S has an infimum. 3

(b) Show that the subset $S=\{x\in\mathbb{Q}\ :\ x>0\,,\ x^2<2\}$ is a non empty subset of $\mathbb{Q}$, bounded below; but $\inf S$ does not belong to $\mathbb{Q}$. 5

3. (a) Show that 0 is a limit point of the set $\{x:\ 0<x<1\}$. 2

(b) Find all limit points of the set of all rational numbers $\mathbb{Q}$. 3

(c) Prove that $\mathbb{Z}$ is not bounded below. 3

4. (a) Prove that the set of all open intervals having rational end points is enumerable. 4

(b) Show that the sequence $\left\{\frac{n^2+2022}{n^2}\right\}$ converges to 1. 4

5. (a) Show that the sequence $\{x_n\}$ is monotone increasing, where 4

$$x_n = 1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}$$

Hence show that the sequence $\{x_n\}$ is not convergent.

(b) Apply Cauchy's criterion for convergence to show that the sequence $\{x_n\}$ is convergent, where 4

$$x_n = 1 + \frac{1}{2!} + \ldots\ldots + \frac{1}{n!} \quad \forall\ n \in \mathbb{N}$$

6. (a) Let $x \in \mathbb{R}$. Show that the series $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ converges absolutely. 4

(b) Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$. 4

7. (a) Discuss the convergence of the series $\sum 1/n^p$, $p > 0$. 4

(b) Let $f_n(x) = x^n$, $x \in [0, 1]$. Show that the sequence of function $\{f_n\}$ is not uniformly convergent on [0, 1]. 4

8. (a) Let $f_n(x) = nxe^{-nx^2}$, $x \in [0, 1]$, $n \in \mathbb{N}$. Show that the sequence $\{f_n\}$ is not uniformly convergent on [0, 1]. 4

(b) Prove that the series $\sum \frac{x}{n + n^2x^2}$ is uniformly convergent for all real x. 4

9. (a) Show that the series $x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots\ldots$ is not uniformly convergent on [0, 1]. 4

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly continuous on $\mathbb{R}$. For each $n \in \mathbb{N}$, let $f_n(x) = f\left(x + \frac{1}{n}\right)$, $x \in \mathbb{R}$. Prove that the sequence $\{f_n\}$ is uniformly convergent on $\mathbb{R}$. 4

10.(a) If $\{u_n\}$ be a sequence of real numbers and $\sum u_n^2$ is convergent prove that $\sum \frac{u_n}{n}$ is absolutely convergent. 4

(b) If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences then prove that, 4

(i) $\{x_n + y_n\}$ is a Cauchy sequence

(ii) $\{x_n y_n\}$ is a Cauchy sequence.

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WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 3rd Semester Examination, 2020, held in 2021

MTMHGEC03T/MTMGCOR03T-MATHEMATICS (GE3/DSC3)

REAL ANALYSIS

Time Allotted: 2 Hours Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Let $A \subseteq \mathbb{R}$ be a non empty set. When is A said to be bounded above? What do you mean by the least upper bound of A ?

(b) Give an example of a bounded above subset E of $\mathbb{R}$ for which $\sup E$ is not a limit point of E.

(c) Show that the sequence $\left\{\frac{3n+1}{n+1}\right\}$ is bounded.

(d) Examine the convergence of the sequence $\left\{\left(\frac{4}{5}\right)^n\right\}$.

(e) Show that the series $\sum_{n=1}^{\infty} a_n$ converges, where

$$a_n = \frac{2n+3}{2n\,(n+1)\,(n+3)}\ ,\ \forall\, n \in \mathbb{N}$$

(f) Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$.

(g) Examine whether the sequence of functions $\{f_n\}$ converges uniformly on $\mathbb{R}$, where for all $n \in \mathbb{N}$,

$$f_n(x) = \frac{x^2}{n}\ ,\ x \in \mathbb{R}.$$

(h) Show that the series $\sum_{n=1}^{\infty} \frac{\cos x^2}{5n^6}$ is uniformly convergent on $\mathbb{R}$.

(i) Is $\sum_{n=1}^{\infty} 2^{-n}\cos(3^n x)$ a continuous function on $\mathbb{R}$? Justify your answer.

(j) Find the radius of convergence of the power series

$$x+\frac{(2!)^2}{4!}x^2+\frac{(3!)^2}{6!}x^3+\cdots\cdots$$

2. (a) Find the least upper bound and greatest lower bound of 2+2

$$S=\{x\in\mathbb{R}\ :\ 3x^2-10x+3<0\}$$

(b) Let S be a non empty bounded subset of $\mathbb{R}$ and let T be a non empty subset of S. Show T is a bounded subset of $\mathbb{R}$. Further show that 2+2

$$\inf S\le\inf T \quad \text{and} \quad \sup T\le\sup S$$

3. (a) State and prove the Archimedean property of $\mathbb{R}$. 1+3

(b) Find the least upper bound and greatest lower bound of 2+2

$$S=\left\{\frac{3}{2},\ -\frac{4}{3},\ \frac{5}{4},\ -\frac{6}{5},\ \frac{7}{6},\ -\frac{8}{7},\ldots\right\}$$

4. (a) Justify that the set of integers $\mathbb{Z}$ has no cluster point. 2

(b) Justify that a finite subset of $\mathbb{R}$ has no cluster point. 2

(c) Show that 1 and –1 are limit points of the set 4

$$S=\left\{(-1)^m+\frac{1}{n}\ ;\ m,\ n\in\mathbb{N}\right\}$$

5. (a) Define a bijective map $f:\mathbb{N}\longrightarrow\mathbb{Z}$ to show that $\mathbb{Z}$ is countably infinite. Justify your answer. 4

(b) Show that [0, 1] is an uncountable set. 4

6. (a) Prove that limit of a convergent sequence is unique. 4

(b) Let $x>0$. Prove that $\lim_{n\to\infty} x^{1/n}=1$. 4

7. (a) Examine the monotonicity of the sequence $\{x_n\}$, where 2+2+1

$$x_n=\frac{2n-1}{3n+4} \quad \text{for all} \quad n\in\mathbb{N}.$$

Hence determine the convergence of $\{x_n\}$. If the sequence $\{x_n\}$ converges, find its limit.

(b) Use Cauchy's criterion for convergence to show that the sequence $\left\{\dfrac{n+1}{n}\right\}$ is convergent. 3

8. (a) Let $x \in \mathbb{R}$. Show that $\sum\limits_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$ if $|x| < 1$. 5

(b) Examine the convergence of the series $\sum\limits_{n=1}^{\infty} \left(\dfrac{n}{2n+1}\right)^n$. 3

9. (a) Show that if the series $\sum\limits_{k=1}^{\infty} a_k$ is absolutely convergent, then $\sum\limits_{k=1}^{\infty} a_k$ is convergent. Give an example, with justifications, to show that the converse may not be true. 4+2

(b) Examine the convergence of the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^2}$. 2

10.(a) Show that the sequence of functions $\{f_n\}$, where for all $n \in \mathbb{N}$, 4

$$f_n(x) = \frac{n^2x^2}{1+n^3x^3} \quad , \quad x \geq 0$$

is pointwise convergent on $[0, \infty)$ but is not uniformly convergent on $[0, \infty)$.

(b) Show that the sequence of functions $\{f_n\}$, where for all $n \in \mathbb{N}$, 4

$$f_n(x) = \begin{cases} nx & ; \quad 0 \leq x \leq \dfrac{1}{n} \\ 1 & ; \quad \dfrac{1}{n} < x \leq 1 \end{cases}$$

is pointwise convergent on [0, 1] but is not uniformly convergent on [0, 1].

11.(a) Show that the series $\sum\limits_{n=1}^{\infty} \dfrac{x}{n(1+nx^2)}$ is uniformly convergent on $\mathbb{R}$. 3

(b) Show that the series $\sum\limits_{n=1}^{\infty} f_n(x)$ where 5

$$f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2} \quad , \quad x \in [0, 1]$$

is not uniformly convergent on [0, 1] but it can still be integrated term by term over [0, 1].

12.(a) Show that the series $\sum_{n=0}^{\infty}(1-x)x^n$ is not uniformly convergent on [0, 1]. 3

(b) Show that the series $\sum_{n=1}^{\infty}\frac{\sin nx}{n^3}$ converges uniformly for all real values of x. Further, if $f(x)$ is the sum function of this series, then show that $f'(x)=\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ for all $x\in\mathbb{R}$. 5

13.(a) Find the radius of convergence of the power series 3

$$\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^n$$

(b) Use the fact that 5

$$\frac{1}{\sqrt{1-x^2}}=1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^{2n}\ ,\ \forall\ |x|<1$$

to obtain the power series of $\sin^{-1}(x)$.

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WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 3rd Semester Examination, 2019

MTMHGEC03T/MTMGCOR03T-MATHEMATICS (GE3/DSC3)

REAL ANALYSIS

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times 5 = 10$

(a) Express real line in terms of a set.

(b) Justify that every real number is a cluster point of $\mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers.

(c) Show that every bounded sequence is not convergent.

(d) Show that pointwise convergence may not imply uniform convergence.

(e) Test the convergence of the series $\sum_{n=1}^{\infty}\frac{n}{2^n}$.

(f) Find the limit function $f(x)$ of the sequence $\{f_n\}$ where

$$f_n(x)=\frac{nx}{1+nx},\quad x\ge 0$$

(g) Use Weierstrass' M-test to show that the series

$$\sum_{n=1}^{\infty}\frac{n^5+1}{n^7+3}\left(\frac{x}{2}\right)^n$$

converges uniformly in $[-2, 2]$

(h) Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{n^n x^n}{n!}$$

2. (a) State and prove Archimedean property of $\mathbb{R}$. 4

(b) Let A be a non empty bounded above subset of $\mathbb{R}$. Let 4

$$-A=\{-x:\ x\in A\}.$$

Show that $-A$ is a non empty bounded below subset of $\mathbb{R}$ and $\inf(-A)=-\sup A$.

3. (a) Show that $\mathbb{N}$ is unbounded above. 3

(b) Prove that the open interval $(0, 1)$ is uncountable. 5

4. (a) Does every infinite subset of real numbers have at least one cluster point? Justify your answer. 2

(b) Does every bounded subset of real numbers have at least one cluster point? Justify your answer. 2

(c) Find the cluster points of the set 4

$$S=\left\{ (-1)^{n+1}\frac{n+2}{n+1} : n\in\mathbb{N} \right\}$$

5. (a) Show that the sequence $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$ is a monotone decreasing sequence and find its limit. 4

(b) Show that $\lim\limits_{n\to\infty} x_n = 1$, where 4

$$x_n=\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}},\ \forall n\in\mathbb{N}$$

6. (a) Test the convergence of the series 4

$$\sum_{n=1}^{\infty}\frac{n^2-1}{n^2+1}\,x^n, \text{ where } x\neq 1.$$

(b) Test the convergence of the series 4

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{2n+1}{n(n+1)}$$

7. (a) State and prove Cauchy's first theorem. 4

(b) Find the limit function $f(x)$ of the sequence $\{f_n\}$ where for all $n\in\mathbb{N}$, 4

$$f_n(x)=\frac{nx}{1+n^2x^2},\ 0\le x\le 1$$

Also show that the sequence $\{f_n(x)\}$ is not uniformly convergent on $[0,1]$.

8. (a) Use Cauchy's general principle of convergence to show that the sequence $\left\{\frac{n}{n+1}\right\}$ is convergent. 4

(b) Find the sum function of the series 2+2

$$x^4+\frac{x^4}{1+x^4}+\frac{x^4}{(1+x^4)^2}+\frac{x^4}{(1+x^4)^3}+\cdots$$

where $0\le x\le 1$. Hence state with reason whether the series is uniformly convergent on $[0,1]$.

9. (a) Find the radius of convergence of the power series 4

$$\sum_{n=0}^{\infty}[3+(-1)^n]^n x^n$$

(b) Assuming the power series expansion 4

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \text{ for } |x|<1,$$

show that $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\cdots;\quad |x|<1.$

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